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The coisotropic subgroup structure of $SL_q(2, \mathbf{R})$

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Abstract

We study the coisotropic subgroup structure of standard $SL_q(2, \mathbf{R})$ and the corresponding embeddable quantum homogeneous spaces. While the subgroups $\mathbf{S}^1$ and $\mathbf{R}_+$ survive undeformed in the quantization as coalgebras, we show that $\mathbf{R}$ is deformed to a family of quantum coisotropic subgroups whose coalgebra cannot be extended to an Hopf algebra. We explicitly describe the quantum homogeneous spaces and their double cosets. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Quantum groups are a natural framework for defining principal bundles and homogeneous spaces in non-commutative geometry. These deeply connected notions have been largely investigated in the recent years and have had a parallel development. The first and best known example of quantum homogeneous spaces is provided by quantum spheres [12]: this is a family of embeddable $SU_q(2)$-comodule algebras. The identification of an appropriate quotient procedure has been more elaborated. Indeed, once one has a *quantum subgroup*, i.e. a quotient by a Hopf ideal, a quantum homogeneous space can be defined as the subalgebra of coinvariant elements. Nevertheless, it is well known (see e.g. [8,13]) that in the quantum case the limited number of subgroups that survive is not enough to obtain all the homogeneous spaces. The concept of quantum subgroup must be generalized to that of *coisotropic quantum*

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subgroup, allowing to recover from a quotient type procedure all known families of quantum embeddable homogeneous spaces. Coisotropic subgroups are quotient by a coideal, right (or left) ideal so that they inherit only the coalgebra structure while the algebra is weakened to a right (or left) module. The name coisotropic that we use emphasizes the semiclassical properties: indeed they are quantizations of coisotropic subgroups of Poisson–Lie groups [6].

In a parallel way quantum principal bundles, together with the theory of connections, were studied in [3] with a Hopf algebra on the fibre. This construction is known in Hopf algebra theory as Galois extension (see e.g. [10]). More general quantum structure groups were considered first in [2] and then in [4]: the total space is an algebra and the structure group is just a coalgebra. According to this definition embeddable homogeneous spaces realize quantum principal bundles with coisotropic subgroups as structure groups.

This construction has been studied for quantum spheres in [2] and for quantum planes and cylinders in [6]. In both cases it comes out that the quantum stability subgroups, which by construction are only coalgebras, can be completed with an algebra structure thus obtaining the undeformed Hopf algebra of the classical stability subgroup.

We show in this paper that the subgroup structure of standard $SL_q(2,\mathbf{R})$, displays a pure quantum behaviour. Besides the deformation of the upper complex plane and the one sheeted hyperboloid, whose stability subgroups remain classical, we exhibit a family of homogeneous spaces whose stability subgroups are only coalgebras not supporting a compatible Hopf algebra.

The real structure, that was not considered in the previous works, plays here a fundamental role. Although the involution does not descend to the coisotropic subgroup, nevertheless the real structure survives by asking that the defining ideal is $\tau = * \circ S$ invariant [1]. Indeed, according to this structure, we are able to describe the different nature of coisotropic subgroups of $SL_q(2,\mathbf{R})$ with respect to those defined by $SU_q(2)$ fibrations over quantum-spheres.

2. Summary of the results

In this section, we describe the results of the paper. Let us first recall the classical case. A geometrical picture can be given by considering the adjoint action of $SL(2,\mathbf{R})$ on its Lie algebra $sl(2,\mathbf{R})$. The Killing form, invariant with respect to the adjoint action, has signature $(1,2)$. After the identification of $sl(2,\mathbf{R})$ with $\mathbf{R}^3$, one obtains that the group action preserves all quadratic submanifolds of the form $x^2 - y^2 - z^2 = \Theta$ with $\Theta \in \mathbf{R}$. Excluding the trivial orbit $(0,0,0)$ and studying the isotropy subgroups of points of such quadrics, one can distinguish three essentially different cases of homogeneous spaces

1. When $\Theta < 0$ one gets one-sheeted hyperboloids; the corresponding isotropy subgroups are all conjugated to

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda \in \mathbf{R}_+ \right\}.$$

These subgroups can also be characterized as those containing hyperbolic matrices, i.e. matrices M such that $|\mathrm{tr}\, M| > 2$. We will call such subgroups to be of $\mathbf{R}_+$-type.

2. When $\Theta > 0$ one gets two sheeted hyperboloids. The connected components of the isotropy subgroups are conjugated to

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in [0, 2\pi) \right\}.$$

These subgroups contain elliptic matrices, i.e. matrices M such that $|\mathrm{tr}\, M| < 2$. We will call such subgroups to be of $\mathbf{S}^1$-type.

3. Finally, when $\Theta = 0$ one gets the light-cone minus its vertex. The connected components of the isotropy subgroups are conjugated to

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, b \in \mathbf{R} \right\}.$$

All of its elements are parabolic, i.e. matrices M such that $|\mathrm{tr}\, M| = 2$. We will call such subgroups to be of $\mathbf{R}$-type.

We remark that another presentation of the homogeneous spaces corresponding to $\mathbf{S}^1$ is the upper complex plane with the homographic action. All one-dimensional subgroups of $SL(2, \mathbf{R})$ belong to one of the conjugacy classes of the isotropy subgroups above. The representatives we have chosen are those appearing in the Iwasawa decomposition $SL(2, \mathbf{R}) = KAN$.

Consider now the quantum situation. Subgroups of $\mathbf{S}^1$ and $\mathbf{R}_+$ type remain undeformed as coalgebras; the subgroups of $\mathbf{R}$ type are deformed to a family of coisotropic quantum subgroups whose coalgebras are all isomorphic to a coalgebra $\mathbf{R}_q$ which cannot be completed to a Hopf algebra. The corresponding homogeneous spaces are the analogues of the exceptional quantum-spheres parametrized in [12] by $c = c(n)$. Contrary to these exceptional quantum-spheres the spaces of our special series have classical points so that they are embeddable. In Proposition 10, we give a detailed analysis of $\mathbf{R}_q$ showing that it is not cosemisimple. In this way, we have a quantization of any classical connected one-dimensional subgroup of $SL(2, \mathbf{R})$. The general problem of classifying all the coisotropic subgroups, including the discrete ones, is a much more delicate task.

We construct the embeddable quantum homogeneous spaces as the spaces of coinvariant elements and we recover the results given in [9] starting from the semiclassical covariant Poisson structure. By the use of coisotropic subgroups, we are able to define the double cosets and give their explicit description.

3. Coisotropic quantum subgroups and homogeneous spaces

In this section, we recall the definition and the main properties of real coisotropic quantum subgroups as well as their associated homogeneous spaces (see [1,6] for more details). We then discuss the equivalence of coisotropic subgroups determined by the characters of the whole Hopf algebra.

Given a real quantum group $(\mathcal{A}, *)$, we will call *real coisotropic quantum right* (*left*) *subgroup* $(\mathcal{K}, \tau_{\mathcal{K}})$ a coalgebra, right (left) $\mathcal{A}$-module $\mathcal{K}$ such that

1. there exists a surjective linear map $\pi : \mathcal{A} \to \mathcal{K}$, which is a morphism of coalgebras and of $\mathcal{A}$-modules (where $\mathcal{A}$ is considered as a module on itself via multiplication);
2. there exists an antilinear map $\tau_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ such that $\tau_{\mathcal{K}} \circ \pi = \pi \circ \tau$, where $\tau = * \circ S$.

A $*$-Hopf algebra $\mathcal{S}$ is said to be a real quantum subgroup of $\mathcal{A}$ if there exists a $*$-Hopf algebra epimorphism $\pi : \mathcal{A} \to \mathcal{S}$; evidently this is a particular coisotropic subgroup. We remark that a coisotropic quantum subgroup is not in general a $*$-coalgebra but it has only $\tau_{\mathcal{K}}$ defined on it. It is easy to verify that if a coisotropic quantum subgroup is also a $*$-coalgebra and $\pi \circ * = * \circ \pi$, it is then possible to complete the structure so as to have a quantum subgroup.

Coisotropic quantum subgroups are characterized by the following proposition.

Proposition 1. *There exists a bijective correspondence between coisotropic quantum right* (*left*) *subgroups and* τ*-invariant two-sided coideals, right* (*left*) *ideals in* $\mathcal{A}$.

A $*$-algebra B is said to be an embeddable quantum left (right) $\mathcal{A}$-homogeneous space if there exists a coaction $\delta : B \to B \otimes \mathcal{A}$, $(\delta : B \to \mathcal{A} \otimes B)$ and an injective morphism of $*$-algebras $i : B \to \mathcal{A}$ such that $\Delta \circ i = (i \otimes \mathrm{id}) \circ \delta$ $(\Delta \circ i = (\mathrm{id} \otimes i) \circ \delta)$.

A canonical construction for embeddable quantum homogeneous spaces can now be provided.

Proposition 2. *If* $(\mathcal{K}, \tau_{\mathcal{K}})$ *is a right coisotropic quantum subgroup of* $(\mathcal{A}, *)$ *then*

$$B^{\pi} = \{a \in \mathcal{A} | (\mathrm{id} \otimes \pi)\Delta a = a \otimes \pi(1)\}$$

is a right embeddable quantum homogeneous space.

If $(\mathcal{K}, \tau_{\mathcal{K}})$ *is a left coisotropic quantum subgroup of* $(\mathcal{A}, *)$ *then*

$$B_{\pi} = \{a \in \mathcal{A} | (\pi \otimes \mathrm{id})\Delta a = \pi(1) \otimes a\}$$

is a left embeddable quantum homogeneous space.

The correspondence between coisotropic quantum subgroups and embeddable quantum homogeneous spaces is bijective only provided some faithful flatness conditions on the module and comodule structures are satisfied (see [11] for more details).

Let $g : \mathcal{A} \to \mathbf{C}$ be a character, i.e. a $*$-homomorphism of $\mathcal{A}$ in $\mathbf{C}$, let then $\mathrm{Ad}_g x = \sum_{(x)} g(S(x_{(1)}))g(x_{(3)})x_{(2)}$. The following proposition is a straightforward consequence of the fact that Ad_g is an algebra and coalgebra isomorphism and commutes with τ.

Proposition 3. *Let* $r : \mathcal{A} \to \mathcal{K}$ *be the projection that defines the right coisotropic subgroup* $\mathcal{K}$. *Let also* $r_g[x] = r[\mathrm{Ad}_g^{-1} x]$, $x \in \mathcal{A}$. *Then*, $\mathrm{Ker}\, r_g = \mathrm{Ad}_g \, \mathrm{Ker}\, r$ *is a* τ *invariant right ideal and two sided coideal and determines the coisotropic subgroup* $\mathcal{K}_g$. *The corresponding homogeneous space* $B^{r_g} = \mathrm{Ad}_g B^r$ *is isomorphic to* B^r *as left comodule algebra*.

We remark that the two quotient structures $\mathcal{K}$ and $\mathcal{K}_g$ are isomorphic as coalgebras but not as right modules. Indeed, $r_g[x\, f] = r_g[x]\mathrm{Ad}_g f$.

4. Coisotropic subgroups

Let q be a complex number of modulus one, not a root of unity. The function algebra on the quantum group $SL_q(2, \mathbf{R})$ is defined as the unital $*$-algebra generated by four real elements a, b, c, d with relations

$$\begin{aligned} &ab = q\,ba, \quad ac = q\,ca, \quad bc = cb, \quad bd = q\,db, \\ &cd = q\,dc, \quad da - q^{-1}\,bc = 1 = ad - q\,bc. \end{aligned} \tag{1}$$

The Hopf algebra structure, in matrix form, is given by

$$\begin{aligned} &\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \\ &\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}. \end{aligned} \tag{2}$$

A direct application of diamond lemma leads to

Proposition 4. *The elements*

$$\{a^r b^s c^t, b^s c^t d^r,\ r, s, t \in \mathbf{N}\} \tag{3}$$

form a basis of $SL_q(2, \mathbf{R})$ *as a complex vector space*.

It is straightforward to verify that all the characters of $SL_q(2, \mathbf{R})$ are of the form

$$g_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \tag{4}$$

with $\alpha \in \mathbf{R}\backslash\{0\}$.

We now describe the family of quantum coisotropic subgroups of $SL_q(2, \mathbf{R})$.

Proposition 5. *Let* $\mathcal{C}_{\mu\nu}$ *be the linear subspace of* $SL_q(2, \mathbf{R})$ *spanned by* $\{(a - d + 2q^{1/2}\mu b), (q\nu b + c)\}$ *with* $\mu, \nu \in \mathbf{R}$. *Then,* $\mathcal{C}_{\mu\nu}$ *is a* τ*-invariant two sided coideal in* $SL_q(2, \mathbf{R})$.

Proof. From the homomorphism property $\tau(ab) = \tau(a)\tau(b)$, the τ-invariance of $\mathcal{C}_{\mu\nu}$ is proved by verifying that

$$\tau(a - d + 2q^{1/2}\mu b) = -(a - d + 2q^{1/2}\mu b), \qquad \tau(q\nu b + c) = -q^{-1}(q\nu b + c).$$

From

$$\varepsilon(a - d) = \varepsilon(b) = \varepsilon(c) = 0,$$

$$\begin{aligned} \Delta(a - d + 2q^{1/2}\mu b) = {}&(a - d + 2q^{1/2}\mu b) \otimes (a + 2q^{1/2}\mu b) \\ &+ (d - 2q^{1/2}\mu b) \otimes (a - d + 2q^{1/2}\mu b) \\ &+ b \otimes (q\nu b + c) - (q\nu b + c) \otimes b, \end{aligned}$$

$$\Delta(q\nu b + c) = (a + 2q^{1/2}\mu b) \otimes (q\nu b + c) + (q\nu b + c) \otimes (d - 2q^{1/2}\mu b)$$
$$-(a - d + 2q^{1/2}\mu b) \otimes c + c \otimes (a - d + 2q^{1/2}\mu b)$$

it follows that $\mathcal{C}_{\mu\nu}$ is a two-sided coideal. □

Let now $\mathbf{R}_{\mu\nu} = \mathcal{C}_{\mu\nu} SL_q(2, \mathbf{R})$ and $\mathcal{L}_{\mu\nu} = SL_q(2, \mathbf{R})\mathcal{C}_{\mu\nu}$ the right and left ideals generated by $\mathcal{C}_{\mu\nu}$; we call $\mathcal{K}_{\mu\nu} = SL_q(2, \mathbf{R})/\mathcal{R}_{\mu\nu}$, ${}_{\mu\nu}\mathcal{K} = SL_q(2, \mathbf{R})/\mathcal{L}_{\mu\nu}$ and $r_{\mu\nu}$ and $\ell_{\mu\nu}$ the corresponding quotient morphisms. We denote by center dot $(\cdot)$ the action of $SL_q(2, \mathbf{R})$ on the quotients.

In the next section, we will study the coisotropic quantum subgroups defined by the right projection.

Lemma 6. *Let* $w_0 = 1$ *and*

$$w_n = (a + q^{1/2}\chi_\sigma\, b) \cdots (a + q^{|n|-1/2}\chi_\sigma b), \quad n \in \mathbf{Z}\backslash\{0\},$$

where $\sigma = n/|n|$ *and*

$$\chi_\sigma = \mu + \sigma\sqrt{\mu^2 - \nu} := \mu + \sigma \exp\{\tfrac{1}{4}\mathrm{i}\pi(1 - \mathrm{sign}(\mu^2 - \nu))\}\sqrt{|\mu^2 - \nu|}.$$

Then $v_n = r_{\mu\nu}[w_n]$ *is group-like,* $\tau(v_n) = v_n$ *for* $\mu^2 < \nu$ *while* $\tau(v_n) = v_{-n}$ *for* $\mu^2 \geq \nu$.

Proof. From the relations $w_n(a + q^{-n+1/2}\chi_- b) = (a^2 + \nu b^2 + 2q^{-1/2}\mu ab)w_{n-1}$ for $n > 0$ and $w_n(a + q^{n+1/2}\chi_+ b) = (a^2 + \nu b^2 + 2q^{-1/2}\mu ab)w_{n+1}$ for $n < 0$ and the equality $r_{\mu\nu}[a^2 + \nu b^2 + 2q^{-1/2}\mu ab] = r_{\mu\nu}[1]$ we have

$$v_{n+1} = v_n \cdot (a + q^{n+1/2}\chi_+ b), \qquad v_{n-1} = v_n \cdot (a + q^{-n+1/2}\chi_- b), \quad n \in \mathbf{Z}. \tag{5}$$

We first prove, by induction, that the elements v_n are group-like. This is trivially true for $n = 0$. Assume then $\Delta v_n = v_n \otimes v_n$ for $n > 0$. After some algebraic rearrangement, making use of the induction hypothesis and the property

$$r_{\mu\nu}[x] \cdot c = -q\nu r_{\mu\nu}[x] \cdot b, \quad x \in SL_q(2, \mathbf{R}), \tag{6}$$

the equality $\Delta v_{n+1} = v_{n+1} \otimes v_{n+1}$ is reduced to the following relation:

$$v_n \cdot d = v_n \cdot (a + (q^{n+(1/2)}\chi_+ + q^{-n+(1/2)}\chi_-)b), \quad n \in \mathbf{Z}, \tag{7}$$

that can be proved again by induction. The results about τ are obtained using the same procedure. The proof is similar for $n < 0$. □

Remark 7. *Analogously the left quantum subgroup obtained as the image of* $\ell_{\mu\nu}$ *contains the group-like elements* $\ell_{\mu\nu}[\tilde{w}_n]$, *where* $\tilde{w}_0 = 1$ *and*

$$\tilde{w}_n = (a + q^{|n|-(1/2)}\chi_\sigma b) \cdots (a + q^{1/2}\chi_\sigma b), \quad n \in \mathbf{Z}\backslash\{0\}. \tag{8}$$

According to the structures contained in coisotropic subgroups different notions of equivalence can be defined. In the following proposition we classify them as morphisms of coalgebras and of modules and coalgebras.

Proposition 8.

1. *The subgroup* $\mathcal{K}_{\mu\nu}$ *is isomorphic as a coalgebra to* $\mathbf{R}_+$ *if* $\mu^2 > \nu$. *It is isomorphic to* $\mathbf{S}^1$ *if* $\nu > 0$ *and* $\mu/\sqrt{\nu} = \cos\phi_{\mu\nu} < 1$ *for* $\cos^2\phi_{\mu\nu} \neq \cos^2\ell\phi$, *where* $\ell \in \mathbf{Z}$ *and* $q = \mathrm{e}^{\mathrm{i}\phi}$. *The coalgebras corresponding to the special series* $\cos^2\phi_{\mu\nu} = \cos^2\ell\phi$ *are mutually isomorphic*.
2. *Let* $\nu > 0$, $\mu/\sqrt{\nu} = \cos\phi_{\mu\nu} \leq 1$ *and* $\mu_n/\sqrt{\nu} = \cos(\phi_{\mu\nu} + n\phi)$. *Then*, $\mathcal{K}_{\mu\nu}$ *and* $\mathcal{K}_{\mu_n\nu}$ *are isomorphic as coalgebras and modules*.

Proof. We first prove the part (2) of the proposition. From (6) and (7) and the definition of $C_{\mu_n\nu}$ it is straightforward to derive that $\mathrm{Ann}_{\mathcal{K}_{\mu\nu}}(v_n) = \mathrm{Ker}(r_{\mu_n\nu})$. From (5), we see that $v_0 \in v_n \cdot SL_q(2,\mathbf{R})$ for each $n \in \mathbf{Z}$. Since $\mathcal{K}_{\mu\nu}$ is generated as a module by v_0, we conclude that

$$\mathcal{K}_{\mu\nu} = v_n \cdot SL_q(2,\mathbf{R}) \simeq SL_q(2,\mathbf{R})/\mathrm{Ann}_{\mathcal{K}_{\mu\nu}}(v_n) = SL_q(2,\mathbf{R})/\mathrm{Ker}(r_{\mu_n\nu}) = \mathcal{K}_{\mu_n\nu}.$$

The module morphism $i : \mathcal{K}_{\mu\nu} \to \mathcal{K}_{\mu_n\nu}$ is defined by $i(v_n \cdot a) = r_{\mu_n\nu}[a]$ and it is clearly a coalgebra morphism.

We now prove the statement (1). Using Proposition 4 it is easy to show that $\mathcal{K}_{\mu\nu}$ is spanned by elements $r_{\mu\nu}[b^s]$ and $r_{\mu\nu}[ab^s]$, $s \in \mathbf{N}$. From (5), we get

$$\begin{aligned} &q^{1/2}(q^n\chi_+ - q^{-n}\chi_-)v_n \cdot b = v_{n+1} - v_{n-1}, \\ &(q^n\chi_+ - q^{-n}\chi_-)v_n \cdot a = q^n\chi_+ v_{n-1} - q^{-n}\chi_- v_{n+1}. \end{aligned} \tag{9}$$

Assuming $\cos^2\phi_{\mu\nu} \neq \cos^2\ell\phi$, namely $(q^\ell\chi_+ - q^{-\ell}\chi_-) \neq 0$ for $\ell \in \mathbf{Z}$, using a recurrence procedure starting from $s = 0$, we find

$$r_{\mu\nu}[b^s] = \sum_{k=0}^{s} C_k^s \; v_{s-2k}, \tag{10}$$

where

$$C_k^s = (-)^k q^{-s/2} \frac{[s]_q!}{[k]_q![s-k]_q!} \prod_{\substack{i=0 \\ i \neq s-k}}^{s} \frac{1}{q^{i-k}\chi_+ - q^{-i+k}\chi_-}.$$

Here, we use the standard notation for the q-numbers and q-factorials. This proves, together with $r_{\mu\nu}[ab^s] = q^s r_{\mu\nu}[b^s] \cdot a$ that $\mathcal{K}_{\mu\nu}$ is spanned by the v_n with $n \in \mathbf{Z}$. From Lemma 6, we have that for $\mu^2 > \nu$ the subgroup $\mathcal{K}_{\mu\nu}$ is isomorphic as real coalgebra to $\mathbf{R}_+$ and for $\mu^2 < \nu$ to $\mathbf{S}^1$.

For each character g_α defined in (4), according to Proposition 2, the real coisotropic quantum subgroup $(\mathcal{K}_{\mu\nu})_{g_\alpha}$ is generated by $\mathcal{C}_{\mu_\alpha\nu_\alpha} = \mathrm{Ad}_{g_\alpha}\mathcal{C}_{\mu\nu}$ with $\mu_\alpha = \mu/\alpha^2$ and $\nu_\alpha = \nu/\alpha^4$. By observing that two subgroups of the special series can be connected by the composition of the adjoint map and a morphism introduced in (2), we conclude that they are isomorphic as coalgebras. □

In the following, we give an explicit description of the coalgebra corresponding to the special series $\cos^2\phi_{\mu\nu} = \cos^2\ell\phi$. By using the result of Proposition 8 (2), we can assume $\ell = 0$, i.e. $\mu^2 = \nu$. We denote such coalgebra as $\mathbf{R}_q$.

Lemma 9. *For each $n \geq 1$ let*

$$X_n = \sum_{i=1}^{n} \frac{(q-q^{-1})^{i-1}\, q^{i/2} \mu^i [n-1]_q!}{[i]_q\,[n-i]_q!} v_{n-i} \cdot b^i.$$

Then

$$\Delta X_n = X_n \otimes v_n + v_n \otimes X_n, \qquad \tau(X_n) = -X_n,$$

and $\{v_n\}_{n\in\mathbf{N}} \cup \{X_n\}_{n\geq 1}$ is a linear basis for $\mathbf{R}_q$.

Proof. The coproduct and the τ of X_n can be calculated by induction observing that $X_1 = q^{1/2}\mu v_0 \cdot b$ and using the relation $X_n \cdot (q^{-n}a - q^n d + 2q^{1/2}\mu b) = -[n+1]_q(q-q^{-1})X_{n+1}$.

Let us define, for each $k \geq 0$, a linear mapping $g_k : \mathbf{R}_q \to \mathbf{C}$ such that $g_k(v_n) = \delta_{kn}$. If we suppose that $\sum_k (\alpha_k X_k + \beta_k v_k) = 0$, for some α_k, β_k and we apply $(\mathrm{id} \otimes g_k)\Delta$, we obtain that $\alpha_k X_k \in \mathrm{Span}\{v_l\}_{l\in\mathbf{N}}$. Since X_n cannot be generated by group-like elements, we conclude that $\alpha_k = 0$ and thus $\beta_k = 0$. Therefore $\{v_n, X_n\}$ are linearly independent. Finally, we observe that $\boldsymbol{R}_q = \mathrm{Span}\{v_0 \cdot b^k, v_1 \cdot b^k\}_{k\in\boldsymbol{N}}$. As $v_1 \cdot b^s \in \mathrm{Span}\{v_k\}_{k\leq s+1} \oplus \mathrm{Span}\{v_0 \cdot b^k\}_{k\leq s-1}$ it is sufficient to show that $v_0 \cdot b^s \in \mathrm{Span}\{v_k, X_k\}_{k\leq s}$. This can be done by induction, when the relation

$$q^{1/2}\mu(q^n - q^{-n})X_n \cdot b = \frac{[n+1]_q}{[n]_q} X_{n+1} - \frac{[n-1]_q}{[n]_q} X_{n-1} - q^{1/2}\mu \frac{(q^n + q^{-n})}{[n]_q} v_n \cdot b$$

is used. □

In the following proposition, we summarize the properties of the coalgebra $\mathbf{R}_q$.

Proposition 10.

1. *If $\mathbf{R}_q^{(n)} = \mathrm{Span}\{v_n, X_n\}$ for any fixed $n > 0$ and $\mathbf{R}_q^{(0)} = \mathrm{Span}\{v_0\}$, then $\mathbf{R}_q = \oplus_{n\in\boldsymbol{N}a} \mathbf{R}_q^{(n)}$ as a direct sum of cocommutative coalgebras.*
2. *The only simple subcoalgebras are those generated by v_n, i.e. $\mathbf{R}_q$ is pointed. The corresponding one-dimensional corepresentations are unitary.*
3. *There exists no Hopf algebra isomorphic to $\mathbf{R}_q$ as coalgebra.*

Proof. The point (1) is a direct consequence of Lemma 9.

Using Theorem (8.0.3) of [14], we have that each simple coalgebra must be contained in $\mathbf{R}_q^{(n)}$ for some n. It is clear that the only simple subcoalgebra of $\mathbf{R}_q^{(n)}$ is $\mathrm{Span}\{v_n\}$. This proves the point (2).

Finally, let us suppose that $\mathbf{R}_q$ has a bialgebra structure and let $v_{n_0} = 1$. By Theorem (8.1.1) of [14] the irreducible component $\mathbf{R}_q^{(n_0)}$ of v_{n_0} must be a sub-bialgebra. It is not difficult to see that the only two-dimensional bialgebra is generated by two group-like elements, so that $\mathbf{R}_q^{(n_0)}$ must be one-dimensional, i.e. $n_0 = 0$. By using Theorem (8.1.5) of [14] if the irreducible component of the identity is one-dimensional then $\mathbf{R}_q$ must be isomorphic as an algebra to $\mathrm{Span}\{v_n\}_{n\in\mathbf{N}a}$. This is not true and shows point (3). □

Remark 11 (Classical limit of the special series). *Eq.* (9) *for* $\mu^2 = \nu$ *reads* $v_{n+1} - v_{n-1} = (q-q^{-1})q^{1/2}\mu[n]_q v_n \cdot b$. *As a consequence, we have* $\{v_{2n} - v_0, v_{2n+1} - v_1\} \in (q-q^{-1})\mathbf{R}_q$, *so that in the classical limit all the group-like elements* v_n *collapse into* v_0 *or* v_1 *according to the parity of* n.

5. Homogeneous spaces of $SL_q(2, R)$ and double coset

Starting from a real coisotropic quantum subgroup one has a naturally defined real embeddable quantum homogeneous space

$$B^{r_{\mu\nu}} = \{x \in SL_q(2, \mathbf{R}) : (\mathrm{id} \otimes r_{\mu\nu})\Delta x = x \otimes r_{\mu\nu}[1]\}.$$

Let

$$z_1 = q^{-1/2}(ac + \nu\, bd) + 2\mu bc, \qquad z_2 = c^2 + \nu d^2 + 2\mu q^{-1/2} cd,$$
$$z_3 = a^2 + \nu b^2 + 2\mu q^{-1/2} ab. \tag{11}$$

By direct computation it can be verified that z_i are real elements with relations

$$z_1 z_2 = q^2 z_2 z_1, \qquad z_1 z_3 = q^{-2} z_3 z_1, \qquad z_3 z_2 = \nu + q^2 z_1^2 + 2\mu q z_1 \tag{12}$$

and coproducts

$$\Delta(z_1) = (1 + (q + q^{-1})bc) \otimes z_1 + q^{-1/2} bd \otimes z_2 + q^{-1/2} ac \otimes z_3 + 2\mu bc \otimes 1,$$
$$\Delta(z_2) = q^{-1/2}(q + q^{-1})cd \otimes z_1 + d^2 \otimes z_2 + c^2 \otimes z_3 + 2\mu q^{-1/2} cd \otimes 1,$$
$$\Delta(z_3) = q^{-1/2}(q + q^{-1})ab \otimes z_1 + b^2 \otimes z_2 + a^2 \otimes z_3 + 2\mu q^{-1/2} ab \otimes 1. \tag{13}$$

The following proposition can be proven according to the lines suggested in Proposition 5.4 of [2].

Proposition 12. *The left comodule subalgebra* $B^{r_{\mu\nu}}$ *is generated by* $\{z_i\}_{i=1,2,3}$.

The characters of $B^{r_{\mu\nu}}$ are given by $g(z_1) = 0$, $g(z_2) = \alpha\nu$, $g(z_3) = 1/\alpha$ with $\alpha \in \mathbf{R}\backslash\{0\}$. Using the map $(\mathrm{id} \otimes g) \circ \Delta$, we see that $B^{r_{\mu\nu}}$ and $B^{r_{\mu'\nu'}}$ are isomorphic as left comodule algebras if $\mu' = \mu/\alpha$ and $\nu' = \nu/\alpha^2$ with $\alpha \in \mathbf{R}\backslash\{0\}$. We remark that g is a restriction of a character defined on the whole $SL_q(2, \mathbf{R})$ only for $\alpha > 0$.

Remark 13. *In the classical limit the last of the relations* (12) *reads* $z_3 z_2 = \nu + z_1^2 + 2\mu z_1$; *posing* $z_1 = z - \mu$, $z_2 = x + y$, $z_3 = x - y$ *we get* $x^2 - y^2 - z^2 = \nu - \mu^2$. *Therefore* $\Theta = \nu - \mu^2$ *is the parameter, defined in Section* 1, *that classifies the homogeneous spaces of* $SL(2, \mathbf{R})$.

For all μ, ν the irreducible corepresentations of the subgroup are one-dimensional and defined by $\rho_j(1) = v_j$, where $j \in \mathbf{Z}$ for $\mathbf{R}_+$, $\mathbf{S}^1$ and $j \in \mathbf{N}$ for $\mathbf{R}_q$. We remark that they are unitary only in the case of $\mathbf{S}^1$ and $\mathbf{R}_q$.

Following the scheme of [1,7], we induce respectively right and left corepresentations of the whole quantum group $SL_q(2, \mathbf{R})$ on

$$B_j^{r_{\mu\nu}} = \{x \in SL_q(2, \mathbf{R}) | (\mathrm{id} \otimes r_{\mu\nu})\Delta x = x \otimes v_j\},$$
$${}_jB^{\ell_{\mu\nu}} = \{x \in SL_q(2, \mathbf{R}) | (\ell_{\mu\nu} \otimes \mathrm{id})\Delta x = v_j \otimes x\}.$$

The coaction map is simply the restriction of the coproduct to these spaces.

As $(\ell_{\mu\nu} \otimes \mathrm{id})\Delta : B_j^{r_{\mu\nu}} \to \mathcal{K}_{\mu\nu} \otimes B_j^{r_{\mu\nu}}$, we can also define the left and right corepresentations ${}_jB_k^{\mu\nu} = {}_kB^{\ell_{\mu\nu}} \cap B_j^{r_{\mu\nu}}$. By direct computation, we can give the explicit characterization of the double coset

$${}_oB_o^{\mu\nu} = \mathrm{Span}\{(z_2 + \nu z_3 - 2\mu z_1)^n\}_{n \in \mathbf{N}}.$$

Since $\mathbf{S}^1$ and $\mathbf{R}_+$ are cosemisimple, we can apply Corollary 1.5 from [11]; we then have, for the corresponding values of μ and ν, the following decomposition:

$$SL_q(2, \mathbf{R}) = \bigoplus_{j \in \mathbf{Z}} {}_jB^{\ell_{\mu\nu}} = \bigoplus_{j \in \mathbf{Z}} S(B_j^{r_{\mu\nu}}).$$

Furthermore, $B_j^{r_{\mu\nu}}$ $({}_jB^{\ell_{\mu\nu}})$ is a finitely generated projective $B^{r_{\mu\nu}}$ $(B_{\ell_{\mu\nu}})$-module. This property is usually rephrased by saying that $B_j^{r_{\mu\nu}}$ $({}_jB^{\ell_{\mu\nu}})$ is the space of sections of a quantum line bundle on the quantum homogeneous space. The space further decomposes as

$$SL_q(2, \mathbf{R}) = \bigoplus_{jk} {}_jB_k^{\mu\nu}.$$

The study of these spaces will be given in a forthcoming paper (see [5] for B_1 for the case of the quantum spheres). We remark, however, that no conclusion can be drawn on vector bundles in the case of $\mathbf{R}_q$ via Ref. [11].

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